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Geometroelectrodynamics

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We present an elementary particle model that can be thought of as a unification of certain topological ideas abstracted from the string model and the standard Yang–Mills theory. The basic dynamical entity of the model is a spacelike 3-surface X^3 in some metric space H and is interpreted as a particle. The dynamics of the model is based on two ideas. First the model is formally a Yang–Mills theory on the surface X^4 representing the "orbit(s)" of the particle(s) in H. Secondly the Yang–Mills structure on X^4 is constructed using only the natural geometric structures of the space H by a process which we call induction. It is found that some rather general requirements highly fix the choice of the space H. In fact the minimal model, for which the space H is the product of Minkowski space and the 2-sphere, is obtained by requiring that the symmetry group of the theory is the product of the Poincaré group and the "color group" $SO(3)$. The unique feature of the minimal model is that it affords a purely topological mechanism for quark confinement.

INTRODUCTION

It is generally believed that the concept of gauge invariance plays an essential role in any realistic attempt at particle description. One might, however, argue that something is still missing: the problem of quark confinement in hadron physics is not yet solved. On the other hand there exist certain phenomenological models [string and bag models: Nambu (1970), Johnson (1975)] in which topological concepts seem to have an important role. Again it is believed that the gauge theory approach can produce these models in certain approximation.

In this work a different approach is adopted. We believe that the topological concepts have an independent, not only phenomenological, role in elementary-particle description and try to unify the gauge-theory approach and the topological ideas abstracted from string and bag models.

To make our goal more concrete, observe that in the string model the orbit of the string can be thought as a 2-manifold in Minkowski space M^4. The basic feature of the model is the topological description of both the quark confinement (string is either closed or has two ends) and particle reactions (strings merge together and open strings can join along their boundaries). The basic idea is to generalize this approach. Instead of 2-surfaces we study 4-surfaces in a metric space $H = M^4 \times S$, where S is some compact metric space with spacelike metric. We interpret the 4-surfaces X^4 as "orbits" of 3-manifolds having particle interpretation. The immediate consequences of this basic hypothesis are the following.

(i) The possibility of classifying the particles (in a sufficiently loose sense) by the topology of the representative 3-manifold. A rough classification is obtained by using only the topology of the boundary X^3: the number of boundary components and the topology of the individual boundary component serve as classification tools. The simplest working hypothesis is that different "generations" (Gaillard and Maiani, 1979) correspond to different boundary topologies and that leptons, mesons, baryons, etc. correspond to 3-manifolds with $1, 2, 3$, etc. boundary components.

(ii) Also the topology of H can have an important role in the particle classification. Namely, for the choice $H = M^4 \times S^2$, which turns out to be the minimal choice allowing Poincaré invariance, the homology group $H_2(H)$ is nontrivial being isomorphic to the group of integers. The boundary components of the 3-manifold X^3 can be classified by their homology charges expressing their homology-equivalence classes in $H_2(H)$. The total homology charge of X^3 is identically zero by the very definition of the homology concept. Thus the interpretation of quarks as boundary components carrying nonvanishing homology charges affords an attractive explanation for the nonobservability of free quarks. In fact it will turn out that the attribute "homological" can be replaced by "magnetic" in the dynamics to be constructed.

(iii) The basic interaction vertices can be classified topologically. The basic vertices changing particle number (the so-called connected-sum and boundary-connected-sum vertices) are obtained by a direct generalization from the corresponding vertices in the string model. Also there are reactions changing the "internal state" of the particle: either the purely internal topology of the corresponding 3-manifold changes (with boundaries remaining unchanged) or the topologies of boundary components change (generation mixing) or even their number changes.

The dynamics of the model is constructed so that formally the standard Yang–Mills theory on the manifold $X^4 \subset H$ results. The basic mathematical tool used is the so-called induction procedure. When applied to the metric, spinor structure and vierbein connection of H it yields the required

Yang–Mills structure on the surface X^4 and the Yang–Mills action can be constructed. Now, however, the Yang–Mills field is not the primary dynamical variable, being of completely geometrical origin and being expressible using only the coordinate variables of the space H. Thus the ultimate dynamical objects are the 3-surface X^3 and spinors defined on it. The spinor structure differs from the conventional Dirac spinor structure: the spinors are those of H (e.g., handedness is defined in H) and the Γ matrices are now dynamical variables being essentially the projections of the Γ matrices of H to the surface X^4.

A feature uniquely characterizing the Yang–Mills action and the decomposition $H=M^4\times S$ is the rich vacuum structure of the resulting theory. It will be shown that any 4-surface having at most a one-dimensional projection to S is a vacuum solution to the equations of motion provided that the spinor field vanishes. The existence of the vacuum solutions leads to a somewhat speculative picture of the macroscopic space-time as a "porridge" of vacuum surfaces making the propagation of the long-range interactions possible (compare with the propagation of sound in matter).

The paper is organized as follows: In Section 1 the basic ingredients of the model are introduced. In Section 2 the immediate consequences of the basic assumptions are studied at the topological level (topological classification of particles and basic interaction vertices). In Section 3 the dynamics of the model is constructed. Yang-Mills structure on the surface X^4 is constructed using the geometric structures of the space H and equations of motion together with boundary conditions are derived. In Section 4 the symmetries and conservation laws of the model are studied. It is found that the isometries of H can be represented as spinor transformations provided that some rather restrictive conditions are satisfied by the metric structure of H. Also the discrete symmetries C, P, T are studied and it is found that C becomes a purely geometric transformation in the model. The model is found to be chirally invariant in a generalized sense (handedness defined in H) and a solution to the so-called chiral problem is suggested by the possibility of applying a handedness condition to the spinors. This condition, however, breaks C, P, T and CPT! In Section 5 the vacuum structure of the model is studied in a speculative spirit, and in Section 6 some solutions to the equations of motion are presented: solutions having photon, string (meson), and neutrino interpretations are obtained.

NOTATION

The basic ingredient of the model to be presented is the metric space H having the decomposition $H=M^4\times S$, where M^4 denotes Minkowski space and S some compact metric space, usually the sphere S^n. The coordinates of H, M^4, and S will be denoted by h^k, m^k and s^k, respectively. For the components of the metric tensor analogous notations h_{kl}, m_{kl}, and s_{kl} will be used.

For the Γ matrices the notation Γ_k will be used. These can be expressed using flat-space Γ matrices with the help of the vierbein coefficients:

$$\Gamma_k = \gamma_{\bar{r}} e_k^{\bar{r}} \tag{N1}$$

The covariant constancy requirement for Γ matrices determines the so-called vierbein connection apart from a gauge transformation in $SO(n)$ for n-dimensional Riemannian space:

$$A_k = -D_k e_l^{\bar{r}} e_{\bar{s}}^l \Sigma_{\bar{r}}^{\bar{s}} \tag{N2a}$$

where D_k means the usual covariant derivative and

$$\Sigma_{\bar{n}}^{\bar{m}} = i/2[\gamma^{\bar{m}}, \gamma_{\bar{n}}] \tag{N2b}$$

An important special choice for H is $H = M^4 \times S^2$. In this case the so-called standard coordinates and standard representation for Γ matrices will be used. The spherical coordinates will be denoted by (Θ, Φ). The nonvanishing components of the metric are

$$s_{\Theta\Theta} = -R^2$$

$$s_{\Phi\Phi} = -R^2 \sin^2\Theta \tag{N3}$$

The standard representation for the Γ matrices is defined as

$$\Gamma_k = \gamma_k \otimes 1, \qquad k = 1, \ldots, 4 \tag{N4a}$$

where the matrices γ_k denote the usual Dirac matrices in representation

$$\gamma^0 = 1 \otimes \sigma^3, \qquad \gamma^i = \sigma^i \otimes \sigma^2 \tag{N4b}$$

(The matrices σ_k are the well-known Pauli spin matrices.)

$$\Gamma_\Theta = R\gamma_5 \otimes \sigma_1$$

$$\Gamma_\Phi = R \sin\Theta \gamma_5 \otimes \sigma_2 \tag{N4c}$$

where the matrix γ_5 is the product of Dirac γ matrices

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \tag{N4d}$$

The vierbein connection has the following nonvanishing components when the usual Cartesian coordinates are used for M^4:

$$A_\Theta = -\cos\Theta\,\sigma_3 \tag{N5a}$$

The nonvanishing components of the curvature form of the vierbein connection are

$$F_{\Theta\Phi} = \sin\Theta\,\sigma_3 \tag{N5b}$$

It is easy to verify the covariant constancy of F and also the fact that it is proportional to the area form of S^2. Thus curvature form is invariant under the area-preserving transformations of S^2 (the group which is isomorphic to the canonical group of two-dimensional phase space).

N-dimensional submanifolds of H will be denoted by the symbol X^n: usually $n=4$ or 3. The following basic notations and definitions will be needed:

Coordinates of X^n: x^α.

Metric of X^n: $g_{\alpha\beta} \equiv h_{kl} h^k_\alpha h^l{}_{\beta}$, where the notation h^k_α is used for the partial derivatives of the coordinate variables of H (as vectors of H they are tangent to the surface X^n).

Γ matrices of X^n: $\Gamma_\alpha \equiv \Gamma_k h^k_\alpha$.

Yang–Mills connection of X^n: $A_\alpha \equiv A_k h^k_\alpha$, where A_k denotes the vierbein connection of H.

The projection of the Riemannian connection of H to X^n: $A_\alpha{}^k{}_l = \{{}_m{}^k{}_l\} h^m_\alpha$ defines the covariant derivative for the quantities which are tensors with respect to H. In particular the so-called second fundamental form is defined by the covariant derivatives of the tangent vectors h^k_α

$$H^k_{\alpha\beta} = h^k_{\alpha\beta} - \left\{{}^{\gamma}_{\alpha\beta}\right\} h^k_\gamma + \left\{{}_{l}{}^{k}{}_{m}\right\} h^l_\alpha h^m_\beta \tag{N6}$$

The dual of the Yang–Mills field $F_{\alpha\beta}$ is defined to be

$$F^*_{\alpha\beta} = (-\det g)^{1/2} \varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \tag{N7}$$

1. BASIC ELEMENTS OF THE MODEL

1.1. Some Features of the String Model

The string model (Nambu, 1970; Jacob, 1974) describes a meson as a string moving in Minkowski space M^4. The dynamics of the model is defined by an action which is the area of the "orbit" of the string measured in the metric induced from M^4 to the surface representing the orbit. The ends of the string are interpreted as quarks, and quark confinement is a purely topological phenomenon: a string either has two ends or is closed.

Many-particle states can be represented as sets of strings or rather 1-manifolds in some spacelike 3-surface of M^4 and the transitions between different physical states are mediated by 2-surfaces having the corresponding 1-manifolds as their spacelike boundaries. There are two basic vertices for the transitions illustrated in Figures 1a and 1b. Strings either merge together or join along their boundary components (note that the 2-surface mediating the topology change is completely smooth). At the quantum level the state is specified by a state functional in the set of 1-surfaces (containing arbitrarily many components) in some spacelike 3-surface of M^4, and transition amplitudes are obtained by summing over all 2-surfaces having the prescribed spacelike boundaries and attaching to an individual surface the phase factor $\exp(iS)$, where S is the classical action (the area of the surface) for the surface in question. Note that we can interpret the 2-surfaces as a generalization of ordinary Feyn-mann diagrams: the lines of the ordinary diagram are thickened to 2-manifolds.

The conclusion is that the nicest features of the string model are the description of quarks as the boundary components of a 1-manifold and the description of interactions being related to its topological structure. On the other hand, the basic difficulties of the model are met already at this level: baryons cannot be described in any natural way (strings cannot have three ends) and the internal structure of the quarks (flavors) is not explained by the model. This state of affairs suggests the generalization of the model at the topological level.

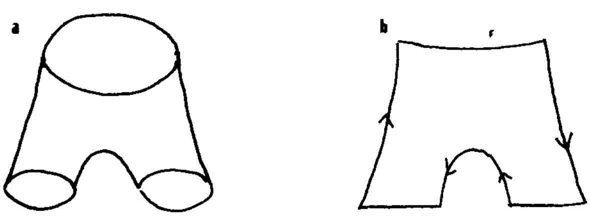

Fig. 1. The basic vertices of the string model. (a) "Trouser vertex." (b) Join along boundaries.

1.2. Generalization of the Topological Structure of the String Model

The most obvious generalization of the string model is to increase the dimension of the basic dynamical entity, i.e., to make it an n-dimensional spacelike submanifold of some "hyperspace" H. Already for $n=2$ we obtain 2-manifolds with arbitrarily many boundary components which we might interpret as quarks. However, the problem of internal structure remains: boundary components have the topology of the 1-sphere S^1. The dimension $n=3$ seems much more interesting: the topology of a single boundary component can be that of a 2-sphere, torus, etc. This dimension also seems to be promising because for $n=4$ the boundary components are 3-manifolds which are too numerous to allow any reasonable physical interpretation. So the basic dynamical entity of the model is taken to be a spacelike 3-manifold imbedded in some metric space H.

The assumption about three-dimensionality of the basic dynamical objects obviously excludes Minkowski space as a candidate for the imbedding space H. Taking as input the requirement of Poincaré invariance, the simplest choice for the imbedding space is $H=M^4\times S$, where S is some compact space with spacelike metric. Taking the scale of S (e.g., the "radius" of S for $S=S^n$) to be of the order of a typical elementary-particle length we can hope that at the macroscopic limit the "transversal" dimensions of H can be neglected, i.e., S can be contracted to a point to a good approximation. The three-dimensionality of the observed world is postulated to reflect simply the three-dimensionality of the observing objects themselves, not the dimension of the space H in which they "live."

1.3. The Dynamics of the Model

The dynamics of the model to be constructed is based on two ideas: First the model should be formally a standard Yang–Mills theory defined on the surface X^4 of H. Secondly the Yang–Mills structure of the model (gauge and spinor fields) should result completely from the geometry of the imbedding space H. The first requirement is inspired mainly by the well-known and successful transition from the Hamiltonian formulation of the classical mechanics to the quantum theory. The second assumption is motivated by the requirement of simplicity and will be realized using the procedure which we call induction: for example, the Yang–Mills connection in X^4 is obtained simply by projecting the vierbein connection of H to the surface. Of course the metric and Yang–Mills connection on the surface X^4 are not the primary dynamical variables: the ultimate dynamical objects are the 3-surface X^3 and the spinor field defined on it.

1.4. The Relation of the Model to the Conventional Theories

We have already spelled out the assumption necessary for any reasonable interpretation of the model: the three-dimensionality of the observed physical world simply reflects the three-dimensionality of the basic objects and thus that of the observer.

From the rough sketch of the model it should be clear that it contains a kind of field–particle duality already at the classical level. At the "short-wavelength limit" corresponding to X^3 with, say, the scale of a typical elementary-particle length, the characterization of X^4 as an orbit of a particle ("Feynmann graph with lines thickened to 4-manifolds") should be appropriate. Approximation of the lines by one-dimensional manifolds should lead to ordinary point-particle dynamics in $M^4 \times S$, and taking into account the transversal scale of S it should be a good approximation to shrink H to M^4. The field aspect should dominate when the scale of X^3 is large enough (for example, long-wavelength photons should satisfy this criterion). At this limit the theory should approach ordinary classical field theory in X^4. Note that when the surface X^4 can be represented as a graph of a map from $M^4 \supset V^4 \to S$ we can speak about a S-valued field in V^4.

2. TOPOLOGICAL LEVEL

2.1. Particle Classification Topologically

(a) Classification Using the Boundary Topology of X^3. The identification of a spacelike 3-manifold of H as a particle leads to the possibility of classifying particles using the topological characteristics of the representative 3-manifold. The roughest classification uses only the topology of the boundary δX^3. The number of the boundary components and the topology of the individual boundary component serve as classification tools.

The topology of an orientable 2-manifold (Wallace, 1968) is characterized by its genus g, i.e., the manifold can be thought of as a sphere with g handles or equivalently as a connected sum of g tori (the connected sum of two n-manifolds (Figure 2) is defined to be the manifold obtained by deleting n-balls from both and by joining the resulting boundaries S^{n-1} by a tube $S^{n-1} \times D^1$). The most general nonorientable closed 2-manifold is expressible as a connected sum of n projective spheres P^2, where P^2 is the manifold obtained by identifying the antipodal points of the sphere S^2. The 2-manifold with several components cannot, however, be a boundary of a 3-manifold unless the "total number" of projective spheres is even (Wallace,

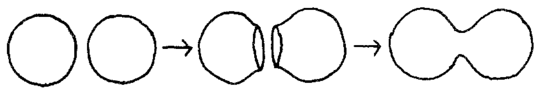

Fig. 2. Connected sum operation for two-dimensional manifolds.

1968):

$$\sum n_k = 0 \bmod 2 \tag{1}$$

The simplest working hypothesis is that different boundary topologies correspond to different particle generations (Gaillard and Maiani, 1979): the multiplets (e, ν_e) and (u, d) have genus $g=0, (\mu, \nu_\mu)$ and (c, s) have $g=1$, etc. The physical interpretation of nonorientable boundary components is left open: it will, however, be seen that in the minimal model $(S=S^2)$ only mesons (strings) made of "orientable" quarks are obtained as classical solutions to the equations of motion. If this working hypothesis is accepted then leptons, mesons, baryons, etc. should correspond to 3-manifolds with $1, 2, 3$, etc. boundary components.

The interpretation of the boundary components as effective particles is also attractive because gauge charges can be defined for these particles as fluxes of the corresponding gauge field components through the boundary component in question by Gauss's theorem.

(b) The Role of the Topology of H in Particle Classification. The topology of the space H affords an exciting possibility to explain the difference between quarks and leptons. It turns out that the minimal model with Poincaré invariance corresponds to the choice $H=M^4 \times S^2$. Now, however, the second homology group $H_2(H)$ (Hilton and Wylie, 1966) is nontrivial, being isomorphic to the group of integers: $H_2(H)=Z$ $(=Z_2$ if nonorientable surfaces are allowed) and we can label each boundary component by an integer expressing the homology equivalence class of the 2-surface in question. However, by the very definition of the homology concept the total homology charge must vanish because the boundary components form together the boundary of a 3-manifold:

$$\sum_k Q_k^H = 0 \; (=0 \bmod 2 \text{ if nonorientable surfaces are allowed}) \tag{2}$$

Thus the physical states are homologically neutral and it is tempting to identify the homologically charged boundary components as quarks. The appearance of a free quark becomes a topological impossibility if this identification is adopted. However, the decay of baryons to leptons might proceed via a kind of "homological depolarization process," as will be seen.

Later it will be shown that the homology charge in fact equals magnetic charge in the dynamics to be constructed. Thus quarks might be called magnetically charged leptons and hadrons are described as magnetic multipoles with quantized pole strengths.

2.2 The Topological Description of Particle Reactions

(a) The General Formulation of the Problem. We are interested in "basic vertices" for the particle reactions interpreted as topology changes of 3-manifolds and the possible topological selection rules. The problem can be formulated as follows:

Given two three-dimensional submanifolds $X^3{}_i$ and $X^3{}_f$ belonging to $(n-1)$-dimensional spacelike submanifolds H_i and H_f of H ($\dim H = n$), respectively, is it possible to find a causal (with locally Minkowskian induced metric) submanifold $X^4{}_{if}$ having $X^3{}_i$ and $X^3{}_f$ as its spacelike boundaries: $\delta X^4{}_{if} \cap H_{i(f)} = X^3_{i(f)}$ (i.e., the 4-manifold in question mediates transitions between initial and final states)? Can we decompose the general transition into more elementary ones, and what are the possible "basic vertices"? Are there any selection rules of topological origin? The problem is well known in topology and is known as the cobordism problem (Wallace, 1968; Thom, 1954; Milnor, 1965).

It is useful to divide the possible particle reactions into the following basic types: (i) The changes of the purely internal topology of the 3-manifold: the number of components of X^3 and the boundary are unaffected. (ii) The reactions changing particle number defined as the number of components in X^3. (iii) The transitions changing the topology of the boundary δX^3: either the topology of an individual boundary component changes or even the number of boundary components changes (string becomes closed).

(b) The Different Reaction Types with Suggested Interpretations. In the following some results for the different transition types will be presented together with physical interpretations suggested by the speculative particle classification adopted.

(i) Changes in Purely Internal Topology. Because in these reactions the topology of the boundary is unchanged it is reasonable to restrict ourselves to the cobordism of the closed (boundaryless) 3-manifolds. One obtains a rough idea about what is involved by noting that the problem reduces to a homology problem (Hilton and Wylie, 1966) if one gives up the requirement that the surfaces are manifolds, which means that they can, for example, intersect themselves. The selection rules for the homology problem result

from the nontriviality of the third homology group $H_3(H)$ (which is trivial, e.g., for $H=M^4\times S^n$, $n\neq 3$).

Thus the possible selection rules result from the requirement that the 4-surface mediating the transition is a causal submanifold of H: the purely internal topology of the 3-manifold and the finite dimension of H can lead to selection rules. It is, however, a well-known result (Thom, 1954) that the so-called abstract cobordism (no imbedding assumed) is trivial for 3-manifolds. The conclusion is that the possible selection rules can result only from the finite dimension of H and from the requirement of causality.

The problem of constructing basic vertices for changes in purely internal topology is solved (Wallace, 1968, Milnor, 1965). The physical interpretation of these reactions suggested by the rough particle classification adopted is that they correspond to changes in "fine structure" not yet observed.

(ii) The Reactions Changing Particle Number. As an immediate generalization of the string-model results we distinguish two types for the reactions changing the component number of the 3-manifold. We call these vertices connected-sum (#) and boundary-connected-sum ($\#_B$) vertices.

The # vertex is a generalization of the "trouser vertex" of the string model and is illustrated in Figure 3 for 2-manifolds. The vertex represents transitions from the disjoint union of 3-manifolds to their connected sum: $A\cup B\rightarrow A\#B$. There are no selection rules involved (in the string model the emission and absorption of a "pomeron" is described by this vertex). It can be shown (Wallace, 1968) that this vertex is essentially the only vertex changing the component number of the n manifold in the cobordism of closed n manifolds. The possible physical processes which one might associate with this vertex are the emission and absorption processes of a pomeron or graviton.

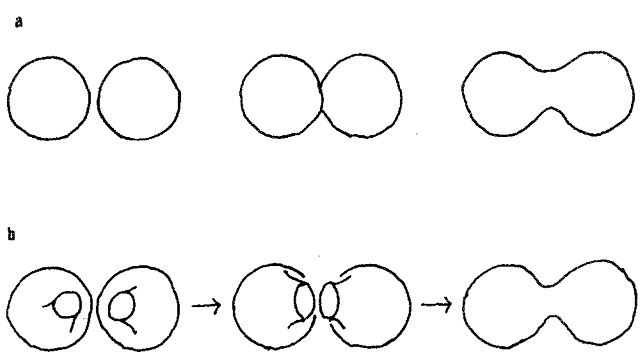

Fig. 3. Basic vertices for two-dimensional manifolds. (a) # vertex; (b) $\#_B$ vertex.

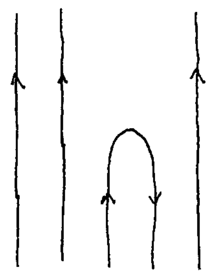

Fig. 4. Quark diagram.

The $\#_B$ vertex represents a transition where two 3-manifolds join along their boundary components. There are obvious selection rules associated with this reaction. The internal topologies of the boundary components must be the same and in case $H=M^4 \times S^2$ the homology charges must be opposite. If the boundary components carry gauge charges they must be opposite also. These reactions are thus described by "quark diagrams" with lines labeled by the values of the homology charge, generation index, and possible other charges. Note that the physical interpretation of the boundary components allows one to speak about flavor conservation in the context of these vertices. (Figure 4 represents an illustration of a quark diagram.)

Note: If the orientability requirement of the surfaces is given up, the requirement that homology charges are opposite in a $\#_B$ vertex is substituted by the requirement that they have the same absolute values.

(iii) The Reactions Changing the Topology of the Boundary. These reactions can be divided into two basic types: those changing the purely internal topology of the boundary component and those changing the number of boundary components.

The reactions of the first type have an interpretation as flavor-changing reactions: they mix different generations (Cabibbo mixing and neutrino oscillations) provided the speculative particle classification is accepted. The problem of possible selection rules is probably that of two-dimensional cobordism, which has only one selection rule: the connected sum of an odd number of P^2s cannot transform to a connected sum of an even number of P^2s or to an orientable manifold (Wallace, 1968).

The reactions changing component number can be divided into two classes: two boundary components either join together (special case of the $\#_B$ vertex: boundary components belonging to the same 3-manifold join together) or they suffer a $\#$ transition ($\#$ vertex for boundaries). Figure 5a illustrates these vertices for two-dimensional manifolds. Note that the decay of a proton might happen through a sequence of the $\#$ transitions if the a

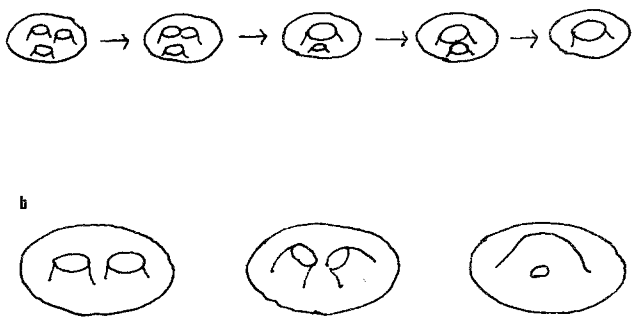

Fig. 5. Changes in the boundary topology. (a) Baryon decay via homological (magnetic) depolarization. (b) Quark annihilation inside hadron (two-dimensional visualization).

boundary components have, for example, homology charges $3, -2, -1$ (Figure 5b).

3. DYNAMICAL LEVEL

3.1. The Induction Procedure

Our aim is to construct a generalization of the string model which is formally the standard Yang–Mills theory on the surface $X^4 \subset H$. Now, however, the Yang–Mills field should be expressible using the coordinate variables of H, which as functions of coordinate variables of X^4 define the surface itself. Thus the ultimate dynamical object is the 4-surface itself. The basic mathematical device used is a process which we call induction. We will induce the metric, the spinor structure, and the Yang–Mills connection to the surface X^4 using corresponding structures of H.

In the case of the metric the induction process simply means the restriction of the line element of H to the surface. Thus the induced metric has components

$$g_{\alpha\beta} = h_{kl} h^k{}_\alpha h^l{}_\beta \tag{3}$$

(where the quantities h^k_α denote partial derivatives of the coordinate variables of H).

In order to induce the spinor structure of H assume that H itself allows spinor structure, i.e., there are globally defined Γ matrices Γ_k satisfying the anticommutation relations (Shanahan, 1978; MTW 1973):

$$\{\Gamma_k, \Gamma_l\} = 2h_{kl} \tag{4}$$

The Γ structure on the surface is defined simply by projecting the Γ matrices of H to the surface

$$\Gamma_\alpha = \Gamma_k h^k_\alpha \tag{5}$$

The obvious requirement

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2g_{\alpha\beta} \tag{6}$$

is satisfied and thus the metric structure is obtained as a by-product of the induction procedure for Γ matrices. The induction of the spinors simply means restriction of the spinors of H to the surface.

The different conjugation-operations, in particular the operation $\Psi \to \overline{\Psi}$ are generalized in an obvious way. The handedness concept is generalized: the spinors are either left- or right-handed with respect to H instead of the 4-surface itself. Note, however, that the handedness concept is defined only for even-dimensional spaces (Shanahan, 1978): only the even-dimensional spaces seem to be physically interesting candidates for the space H. An important feature of the induced spinor structure is that it is defined for all topologies of 4-surface unlike the ordinary spinor structure (Shanahan, 1978). Spinors have $2^{\dim H/2}$ components when H is even dimensional and $2^{\dim H/2-1}$ components when the handedness condition is applied.

The Yang–Mills connection on X^4 results from the canonical vierbein connection of the space H, which in turn is determined modulo a position-dependent rotation in the tangent space from the requirement that Γ matrices are covariantly constant matrices in H (see the Appendix). Thus the vierbein connection A_k is defined by the Riemannian structure of H and has the representation

$$A_k = A_k{}^{mn}\Sigma_{mn} \tag{7a}$$

$$A_k{}^{mn} = \tfrac{1}{4}\{\Gamma^m, D^n\Gamma_k\} \tag{7b}$$

where D^n denotes the covariant derivative with respect to the usual metric connection and Σ_{mn} denotes the spin matrix defined by the commutator of the Γ matrices. The curvature form of the vierbein connection can be expressed using the curvature tensor of H:

$$F_{kl} = \tfrac{1}{2}R_{klmn}\Sigma^{mn} \tag{8}$$

The Yang–Mills connection is defined simply as the projection of the vierbein connection A_k to the 4-surface

$$A_\alpha = A_k h^k_\alpha \tag{9}$$

The curvature form of this connection satisfies the necessary condition

$$F_{\alpha\beta} = F_{kl} h^k_\alpha h^l_\beta \tag{10}$$

Also another connection will be needed. Namely, the so-called second fundamental form (Eisenhart, 1964) of the 4-surface is defined using the covariant derivatives of tangent vectors $h^k{}_\alpha$ (interpreted as vectors in H) with respect to the connection obtained by projecting the Riemannian connection of H to the surface:

$$H^k{}_{\alpha\beta} = h^k{}_{\alpha\beta} - \left\{ {\gamma \atop \alpha\beta} \right\} h^k{}_\gamma + \left\{ {k \atop l\ m} \right\} h^l{}_\alpha h^m{}_\beta \equiv D_\alpha h^k{}_\beta \tag{11}$$

In what follows D_α will mean this kind of covariant derivative when applied to the tensors in H.

3.2 Action Principle and Equations of Motion

Using the induced metric, spinor, and Yang–Mills connection on the surface X^4 the model can be defined formally as a standard Yang–Mills theory on the surface X^4. Thus the action of the model is

$$S = \int_{X^4} dx^4 (-\det g)^{1/2} \left\{ \frac{-1}{4g^2} \mathrm{Tr}\left(F^{\alpha\beta} F_{\alpha\beta} \right) + \frac{i}{2} \left(\bar{\Psi} \Gamma^\alpha \vec{D}_\alpha \Psi - \bar{\Psi} \overleftarrow{D}_\alpha \Gamma^\alpha \Psi \right) \right\} \tag{12}$$

The antisymmetrization in the spinor part is necessary for the reality of the action. It will be seen that the model is chirally invariant (in a generalized sense) and thus it is possible to apply the handedness condition on spinors: this induces an obvious change to the action. The action contains two constants when H is of the form $H = M^4 \times S^n$: these are the gauge-coupling constant and the "'radius" of S, which is expected to be some "hadronic" length.

The equations of motion follow from the stationarity requirement of action. The boundary conditions follow from the corresponding requirement for the boundaries (note, however, that the addition of a total divergence to the action changes boundary conditions). The simplest way to derive the equations of motion is perhaps to handle the metric and the connection as independent dynamical variables expressing the defining equations using Lagrange multipliers (the boundary conditions obtained by this procedure will be wrong).

The equations of motion for the spinor field are

$$\Gamma^\alpha \vec{D}_\alpha \Psi = M\Psi \tag{13a}$$

where the matrix M can be expressed using the second fundamental form:

$$M = -\tfrac{1}{2} D_\alpha \Gamma^\alpha = -\tfrac{1}{2}\Gamma_k H^k_{\alpha\beta} g^{\alpha\beta} \tag{13b}$$

The square of this matrix is proportional to identity matrix and the eigenvalues of M are thus

$$m_\pm = \pm\left(h_{kl} B^k B^l\right)^{1/2} \tag{14a}$$

where the vector B^k is defined as

$$B^k = g^{\alpha\beta} H^k_{\alpha\beta} \tag{14b}$$

The matrix M vanishes when the surface is harmonic: $B^k = 0$ (this equation can be thought of as a generalization of the massless Klein–Gordon equation!) Thus the term "mass matrix" is justified. The boundary conditions are

$$n_\alpha \Gamma^\alpha \Psi = 0 \tag{15}$$

and guarantee the conservation of the currents $(1 \pm i\Gamma_{n+1})\Psi$. When the handedness condition $i\Gamma_{n+1}\Psi = \pm\Psi$ is applied, only one of these currents is nonvanishing. Observe that the vector n_α must be lightlike.

The equations of motion for the coordinates h^k of the 4-surface are obtained most easily using Lagrange multipliers, and they have the following form:

$$D_\alpha\left\{T^{\alpha\beta} h^k{}_\beta + \tfrac{1}{4}\bar{\Psi}(\Gamma^k \vec{D}^\alpha - \overleftarrow{D}^\alpha \Gamma^k)\Psi\right\} - \mathrm{Tr}\left(j^\alpha F^k{}_r h^r{}_\alpha\right) = 0 \tag{16a}$$

The quantities $T^{\alpha\beta}$ and j^{α} are defined as

$$T^{\alpha\beta}=\mathrm{Tr}\left(F^{\alpha\gamma}F_{\gamma}^{\ \beta}\right)$$
$$+\tfrac{1}{4}g^{\alpha\beta}\mathrm{Tr}\left(F^{\gamma\delta}F_{\gamma\delta}\right)+\tfrac{1}{4}\overline{\Psi}(\Gamma^{\alpha}\vec{D}^{\beta}+\Gamma^{\beta}\vec{D}^{\alpha}-\overleftarrow{D}^{\alpha}\Gamma^{\beta}-\overleftarrow{D}^{\beta}\Gamma^{\alpha})\Psi \tag{16b}$$

$$j^{\alpha}=D_{\beta}F^{\alpha\beta}-\overline{\Psi}\Gamma^{\alpha}\Sigma_{kl}\Psi\Sigma^{kl} \tag{16c}$$

The equations can be thought of as a generalization of equations defining the Lorentz force in electrodynamics. The boundary conditions are obtained from the requirement that the boundaries are stationary:

$$n_{\alpha}\left\{T^{\alpha\beta}h^{k}_{\ \beta}+\tfrac{1}{4}\overline{\Psi}(\Gamma^{k}\vec{D}^{\alpha}-\overleftarrow{D}\Gamma^{k})\Psi-\mathrm{Tr}\left(F^{\alpha\beta}F^{k}_{\ r}h^{r}_{\ \beta}\right)=0\right. \tag{17}$$

The boundary conditions are not, however, unique: they change when a total divergence is added to the action. It is remarkable that the conservation of gauge charge can be guaranteed by adding a total divergence $D_{\alpha}(\Omega j_{G}^{\alpha})$ to the action. Note that the gauge current is identically divergenceless, and in the quantization using the functional integral the addition of the total divergence (Ω is of course interpreted as an auxiliary variable) guarantees that the gauge charge is conserved for all 4-surfaces contributing to the functional integral.

4. SYMMETRIES AND CONSERVATION LAWS

4.1. The Representation of the Isometries of H

A quite natural expectation is that the isometries of H should be symmetries of the model. One problem to be solved is the action of the isometries on the spinors on H. Also one must show that the action defining the model is indeed invariant under these transformations. The first problem is not trivial (Shanahan, 1978): although the isometries might have an infinitesimal representation, they need not be representable globally.

We shall approach the above-mentioned problems from the "infinitesimal" point of view. So let the vector j^{k} denote the infinitesimal generator of an isometry ($h^{k}\rightarrow h^{k}+\varepsilon j^{k}$) satisfying thus the well-known Killing identities (MTW, 1973)

$$D_{k}j_{l}+D_{l}j_{k}=0 \tag{18}$$

A straightforward generalization from the case of Minkowski-space spinors would be the transformation formula

$$\delta\Psi = (i\varepsilon/2) D_l j_k \Sigma^{kl} \Psi \tag{19}$$

This transformation formula cannot, however, be the correct one because the corresponding Noether current would not be a gauge-invariant quantity containing a term of the form $\bar{\Psi}\Gamma^{\alpha}A_k\Psi$. The simplest generalization leading to a gauge-invariant current is the transformation formula

$$\delta\Psi = i\varepsilon\left(D_l j_k (\Sigma^{kl}/2) + j^k A_k\right)\Psi \equiv iX\Psi \tag{20}$$

The correction term has a simple interpretation: the isometry is interpreted as a flow in H and the spinor field is translated along the flow lines using parallel translation defined by the vierbein connection A_k. Thus the spinor field suffers a gauge transformation given by the so-called nonintegrable phase factor along the flow line (Wu and Yang, 1976).

The problem is whether this ansatz really works, and we shall prove the following theorem:

> *Theorem.* Let the space H be a product of constant-curvature spaces and two-dimensional manifolds. Then the quantity $\bar{\Psi}\Gamma^{\alpha}D_{\alpha}\Psi$ is invariant under the isometries of H realized according to (20).

Proof. The metric of X^4 is invariant under the action of isometries of H and thus it is sufficient to consider the term $L_{\alpha\beta} \equiv \bar{\Psi}\Gamma_{\alpha}D_{\beta}\Psi$. The change of this quantity can be written in the following form:

$$\delta L_{\alpha\beta} = \bar{\Psi}(K_{\alpha}D_{\beta} + \Gamma_{\alpha}L_{\beta})\Psi \tag{21a}$$

where the quantities K_{α} and L_{α} are defined as

$$\begin{aligned} K_{\alpha} &= \delta\Gamma_{\alpha} + \varepsilon[\Gamma_{\alpha}, X] \\ L_{\alpha} &= \delta A_{\alpha} + \varepsilon[A_{\alpha}, X] \end{aligned} \tag{21b}$$

Obviously the requirements $K_{\alpha}=0$ and $L_{\alpha}=0$ guarantee the invariance of the action. The isometry would thus act like a gauge transformation. These conditions can be transformed in a simpler form by using the definitions of connection and Γ matrices

$$\Gamma_{kr} j^r + [\Gamma_k, X] + \Gamma_r j^r{}_k = 0 \tag{22a}$$

$$A_{kr} j^r + A_r j^r{}_k + X_k + [A_k, X] = 0 \tag{22b}$$

(remember the short-hand notation for the partial derivatives).

The first equation is identically true, as is seen by using the covariant constancy of the Γ matrices and Killing identities (MTW).

The derivatives of the infinitesimal generator of symmetry can be eliminated using the identity defining the curvature tensor

$$D_m D_n j_k - D_n D_m j_k = R^s{}_{kmn} j_s \tag{23}$$

and Killing identities. Using the representation of F_{kr} in terms of the curvature tensor of H the second equation can be cast into the form

$$j^r\left(R_{rmkn} + \tfrac{1}{2}R_{krmn}\right)\Sigma^{mn} = 0 \tag{24}$$

It is easy to show that this equation is satisfied when the curvature tensor of H has the form

$$R_{klmn} = k\left(h_{km}h_{ln} - h_{kn}h_{lm}\right) \tag{25}$$

or more generally H is a product of spaces with curvature tensors satisfying (25). It is a well-known result that this condition is identically satisfied for two-dimensional manifolds, and for higher dimensions the condition implies that the space is a constant-curvature space (Eisenhart, 1964). Thus the representability requirement of isometries seems to imply constraints on the choice of possible symmetry structures of H.

The spaces allowing the representation of isometries by parallel translation have also the property that the "instanton density" (Polyakov, 1975) $\mathrm{Tr}(F^{\alpha\beta}F^*_{\alpha\beta})$ vanishes identically for them, as is seen by using the expression for the curvature tensor of H. In case $S=S^2$ this means that electric and magnetic fields are identically orthogonal and thus the microscopic gauge field is essentially nonlinear.

4.2. Isometry Charges in the Minimal Model

The minimal model affords a topological confinement mechanism and thus the symmetry structure of the model is especially interesting. The isometries of $S=S^2$ form the group $SO(3)$ and the problem is whether we can identify this group as the "weak group" $SU(2)_W$ or should we adopt the interpretation "color group" (Gell-Mann, 1979).

In order to find the answer to this question we must study the transformation properties of the variables $g_{\alpha\beta}$, $F_{\alpha\beta}$, and Ψ under the isometries. The components of the metric are certainly singlets under $SO(3)$ and the same holds for the components of the gauge field because it is proportional to the area form of S^2 induced to X^4 and is thus invariant under the $SO(3)$ transformations [in fact the $SO(3)$ transformations are represented as gauge transformations]. It is rather surprising that the spinor field also behaves like an $SO(3)$ singlet; $SO(3)$ transformations are represented as $U(1)$ gauge transformations, which do not mix the components, and in particular the rotations $\Phi \to \Phi + \varepsilon$ are represented trivially in the "standard gauge"! The fact that $SO(3)$ transformations are represented as pure gauge transformations suggests that they should not change the physical states in any observable way and so these should be $SO(3)$ singlets. Therefore the interpretation of $SO(3)$ as a "colour group" seems to be necessary.

An important question is whether one can attach $SO(3)$ charges to the quarks. By studying the expression for a $SO(3)$ current

$$j^{\alpha} = T^{\alpha\beta} h^{k}{}_{\beta} h_{kl} j^{l} + \tfrac{1}{4}\overline{\Psi}\left(\Gamma_k \vec{D}^{\alpha} - \overleftarrow{D}^{\alpha}\Gamma_k\right)\Psi j^{k}$$

$$+ \tfrac{1}{4}\overline{\Psi}\left(\left\{ j_{kl}\Sigma^{kl}, \Gamma^{\alpha}\right\}\right)\Psi + F^{\alpha\beta}F_{kl}s^{k}{}_{\beta} j^{l} \tag{26}$$

it is seen that it contains a part expressible as a divergence. This term gives to the charge a contribution expressible as a flux through the boundary component and it is natural to define the $SO(3)$ charges of the quarks by this flux factor. For example, for the rotation $\Phi \to \Phi + \varepsilon$ this flux factor is

$$Q^{3}{}_{i} = \int_{\delta X_i^3} F^{*}_{\alpha\beta} \cos\theta \, dx^{\alpha} \wedge dx^{\beta} \tag{27}$$

4.3. Chiral Symmetry

Besides being invariant under the phase multiplication

$$e^{i\alpha}\Psi$$

the action of the model is also invariant under the chiral transformation

$$e^{i\alpha\Gamma}n + 1\Psi$$

when H is even dimensional. This invariance is the generalization of the chiral invariance of the massless field theories. The possibility of choosing the handedness of the spinors suggests an attractive solution to the peculiarities of the conventional chiral symmetry [the approximate nature of the symmetry, the absence of the parity doubling, (Mahantappa and Randa, 1980)]. In particular the absence of the parity doubling might result simply from the handedness condition. Note that the minimal model with the handedness condition is formally almost equivalent to spinor electrodynamics and allows the possibility of the homological quark confinement.

4.4. Discrete Symmetries

It is of substantial interest to find the generalizations of the discrete symmetries C, P, and T of the conventional field theory (Bjorken and Drell, 1965). We shall restrict ourselves to the case $S=S^2$, constructing first the symmetries of the theory without the handedness condition and after that study the symmetry-breaking effects induced by the handedness requirement.

(a) Reflection *P*. A simple guess motivated by the transformation formula of the Dirac spinors is given by

$$m^k \to P(m^k)$$

$$\Psi \to P\Psi \tag{28}$$

where the matrix P has the form

$$P=\gamma^{\hat{0}} \times \sigma^3 \tag{29}$$

in the standard representation of Γ matrices (see Notations). One can verify that the action is indeed invariant under the proposed transformation.

(b) Charge Conjugation *C*. The charge-conjugation operation changes the sign of the connection A. This can be achieved by performing a reflection in S^2

$$\Theta \to \pi - \Theta$$

$$\Phi \to \Phi + \pi \tag{30}$$

(standard coordinates used for S^2) besides the usual C operation for the spinor field, which in the standard gauge has the form

$$\Psi \to C\Psi \tag{31}$$

with C equal to

$$C = i\gamma^2 \times \sigma^3 \tag{32}$$

It is a straightforward task to verify that the equations of motion are transformed to their complex conjugates in the transformation and thus C is a symmetry. A nice feature of the model is that the charge-conjugation operation has a purely geometric interpretation, being analogous to the time-reflection symmetry.

(c) Time Reflection *T*. The generalization of the T symmetry is given by the transformation formula

$$m^k \to T(m^k)$$

$$\Psi \to T\Psi \tag{33}$$

where the matrix T has the form

$$T = i\gamma^1\gamma^3 \otimes \sigma^1 \tag{34}$$

in the standard representation. The action is transformed to its complex conjugate and is thus invariant (being real).

The conclusion is that the model without the handedness condition allows the conventional discrete symmetries allowing a geometric operation for the C operation. When the handedness condition is applied, only those symmetries which do not change handedness rėmain unbroken. Equivalently, only those symmetries having a matrix representation which commutes with the matrix $i\Gamma_7$ remain unbroken. From the representation of these matrices it can be seen that P, C, and also T are broken. The breaking of T symmetry is quite an unexpected result and leads also to the breaking of CPT(!). On the other hand the symmetries CP, TP, and TC remain unbroken in the sense that they leave the action and handedness condition invariant. However, the boundary conditions for the spinor field

$$n^\alpha \Gamma_\alpha \Psi = 0 \tag{35}$$

are not invariant under CP and CT (although invariant under C, P, and T) and thus we can expect symmetry-breaking effects (probably small). Thus classically the symmetry PT is the only exact symmetry.

4.5. Gauge Charge in the Minimal Model

A crucial test for the model is whether it can explain the observed charge pattern of elementary particles. A basic feature seems to be the appearance of the fermions in doublets so that charges within the doublet differ by one unit. Usually the charges $(-1,0)$ and $(2/3,-1/3)$ are attached to the lepton and quark doublets, respectively. There are, however, other alternatives, and perhaps the simplest one is to assume only one basic doublet, say L, carrying charges -1 and 0, respectively, together with its antidoublet $\bar{L}$. Mesons and baryons can be thought of formally as the composites $L\bar{L}$ and $LL\bar{L}$, respectively. It will be seen that this scheme seems to be the most natural one in the minimal model.

In order to approach the charge problem note first that the observed doublet structure cannot be explained group theoretically in the minimal model because the interpretation of the group $SO(3)$ as a color group seems necessary. Also note that the gauge charge can be expressed as the sum of the gauge fluxes through the boundary components using Gauss's theorem, and it is indeed possible to talk about the charge of the quark. Because the gauge field is nonlinear (having vanishing "instanton density") and because it has to satisfy boundary conditions, it is quite reasonable to expect the quantization of the gauge flux already at the classical level.

The simplest scheme is obtained by making the following assumptions. First assume that the gauge flux through a single boundary component obtains the "universal" values 0 or ± 1 (in suitable units). When the homology charge of the boundary component is nonvanishing (the homology charge is in fact the same as magnetic charge), assume that the sign of the gauge charge is determined by the sign of the homology charge, i.e., $Q_G = kQ_H/|Q_H|$, where k is some universal constant. Taking into account the fact that the boundary components of mesons carry opposite homology charges and that baryons (antibaryons) have always 2(1) negatively and 1(2) positively charged boundary components, one finds that the simplest scheme for the charge assignments results: mesons and baryons have effectively $L\bar{L}$ and $LL\bar{L}$ structures, respectively.

A rather satisfactory feature of this scheme is that it explains why the charged leptons appear to have two times more components than the neutral ones. This follows simply from the fact that the leptons L^+ and L^- should be counted as different particles because there is no (discrete) symmetry relating them (CP and CT are broken by the boundary conditions and C is broken by the handedness condition). For quarks the situation is different: there are equally many charged as uncharged quark states because the homology charge can have two values for neutral quarks and because the signs of the gauge charge and magnetic charge are correlated for charged quarks.

Is it really possible to find an ansatz for the boundary behavior of the gauge field consistent with this charge-quantization scheme? The simplest ansatz leading to a trivial gauge charge is the assumption that the normal component of the gauge field vanishes. In fact a family of solutions having interpretation as neutral mesons and satisfying this condition will be obtained. An ansatz leading to a nontrivial gauge charge when the spinor field vanishes is based on the "duality condition"

$$F_{\alpha\beta}=\Omega F^{*}_{\alpha\beta}=\Omega(-\det g)^{1/2}\varepsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} \tag{36}$$

satisfied on the boundary component. Because the "star operation" gives identity when performed twice the coefficient of proportionality can have only the values $\Omega=\pm 1$. As a result the gauge charge is indeed quantized and equal to the magnetic charge of the boundary component. Assuming that the known mesons are composed of quarks with magnetic charge ± 1 their charges could be explained with this ansatz. However, for mesons carrying higher homological charges on their boundaries the unit of charge would be apparently a multiple of the elementary charge.

5. VACUUM STRUCTURE OF THE MODEL

5.1. Vacuum Degeneracy, when $H=M^4\times S$

The Lagrangian of the model is quadratic both with respect to spinor and gauge fields. This implies that all configurations with vanishing gauge and spinor fields satisfy the classical equations of motion. When H has the decomposition $H=M^4\times S$ the gauge group $SO(n+3,1)$ reduces to $SO(n)$ (a highly desirable feature of this choice, because a noncompact gauge group leads, for example, to an energy density which is not positive definite). As a consequence, any surface having at most one-dimensional projection into S is a vacuum solution. To see this choose the coordinates of S so that the solution can be written in the form

$$\begin{aligned} m^k &= m^k(x) \\ s^k &= \text{const}, \qquad k=1,\dots,n-1 \\ s^n &= s^n(x) \end{aligned} \tag{37}$$

The gauge field $F_{\alpha\beta}$ clearly vanishes because it is the projection of F_{kl}, which is an antisymmetric tensor. For example, surfaces of the form $X^3\times S^1\subset M^4\times S$ represent vacua.

The result implies that the model has extremely rich vacuum structure. Any surface X^3 with at most one-dimensional projection to S represents a "vacuum particle" having almost "free will": the only dynamical constraint to its behavior is the requirement that it keeps the projection to S at most one dimensional. Because these solutions have vanishing classical energy-momentum tensor and because only the metric degree of freedom is excited for these solutions, the interpretation of the "vacuum particles" as gravitons is not ruled out.

There are many questions related to the interpretation of these solutions: Is the vacuum of the model a kind of "graviton porridge" making the propagation of the macroscopic long-range interactions possible in a way analogous to the propagation of sound in matter? Can we think of the ordinary space-time as an approximation of this porridge? Could we think the cosmic background as an indication about this vacuum structure? How is the gravitational attraction generated ($G=R^2e^{-1/\alpha}$)?

It deserves to be noted that in the minimal model there might also be vacuum solutions with nonvanishing gauge fields, because for the self-dual fields satisfying $F_{\alpha\beta}=\pm F^*_{\alpha\beta}$ the tensors $T^{\alpha\beta}$ and j^α vanish identically.

Note: It is also possible to construct systems with a finite number of degrees of freedom having "free will" in some part of configuration space. Probably the simplest one is the two-dimensional system characterized by the Lagrangian

$$L=a(x-y)^2(\dot{x}^2+\dot{y}^2)+b(x-y)^2$$

Solutions with $x=y$ have zero energy and momentum and have "free will," i.e., x is an arbitrary function of time. The quantization of the system leads to a system resembling a double potential well, i.e., the ground-state wave function is concentrated around lines situated symmetrically about the line $x=y$.

6. SOLUTIONS TO THE EQUATIONS OF MOTION

In this section some solutions to the equations of motion will be obtained. The solutions have interpretation as photon, neutral meson, and neutrino (electronic and myonic).

6.1. Photon Solution

The photon is conventionally characterized by its polarization and wave vectors satisfying the well-known orthogonality relations. We will

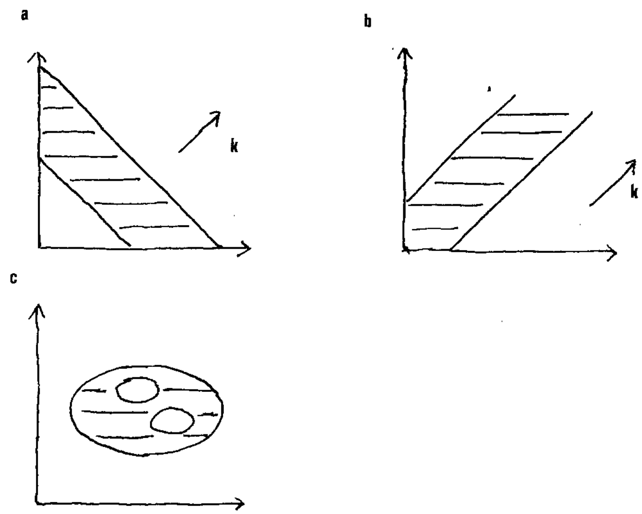

Fig. 6. Illustration of "photon geometry" (a) and (b) possible projections of X^4 to $M_1{}^2$ plane. (c) possible projection of X^4 to $M_2{}^2$ plane.

represent a solution based on an intuitive picture of the photon as a small cylinder moving with the velocity of light along the direction of its axis and carrying electric and magnetic fields orthogonal to its direction of propagation. The 4-surface in $M^4 \times S^2$ corresponding to this intuitive picture can be represented as a graph of an S^2-valued field defined in a region

$$X^2 \times Y^2 \subset M^2{}_1 \times M^2{}_2 = M^4$$

where M_1^2 and M_2^2 are orthogonal linear subspaces of M^4 [for example the (m^1, m^2) and (m^0, m^1) planes of M^4]. X^2 is a disk in M_1^2 with an arbitrary number of holes (Figure 6a) and Y^2 can be thought as a piece of the m^1 axis moving with the velocity of light along this axis (Figure 6b).

The analytic form of the solution using Minkowski coordinates for the surface X^4 is

$$m^k = \delta^k{}_\alpha x^\alpha$$

$$\Theta = f(k \cdot x)$$

$$\Phi = g(k \cdot x, x_T) \tag{38}$$

where k is a lightlike vector in the $M_2{}^2$ plane and x_T denotes the coordinates of M_1^2. The functions are arbitrary. The dependence on arguments is chosen so that the resulting electric and magnetic fields are orthogonal to the direction of motion (note that we can interpret different tensors in X^4 also as tensors in M^4).

The equations of motion (16) are satisfied because the energy-momentum tensor $T^{\alpha\beta}$ and current vector j^α interpreted as M^4 tensors are proportional to $k^\alpha k^\beta$ and k^α, respectively, and the equations of motion involve contractions of the form $k\cdot\nabla_T$ and $k\cdot k$ (in the Minkowski metric) which both vanish (∇_T symbolizes the transversal gradient).

The projection of the boundary of the solution manifold to M^4 has the representation

$$P(\delta X^4)=\delta X^2\times Y^2 U X^2\times\delta Y^2$$

Here P denotes the projection in question. The conservation requirements for momentum and isometry charges lead to the requirements

$$n_\alpha T^{\alpha\beta}=0 \tag{39}$$

$$n_\alpha F^{\alpha\beta}s^k{}_\beta=0 \tag{40}$$

On the part of the boundary corresponding to $\delta X^2\times Y^2$ these conditions are satisfied identically because both boundary conditions involve contractions of the type $k\cdot k$ and $k\cdot\nabla_T$. In the remaining part of the boundary the conditions can be satisfied in two ways. Figure 6b illustrates the case in which the photon moves in the direction of k^α. The boundary condition is satisfied when $F_{\alpha\beta}$ vanishes on the boundary, i.e., the condition

$$k\cdot m=f^{-1}(n\pi) \tag{41}$$

is satisfied on the boundary. The second possibility is illustrated in Figure 6c. Choosing the normal of Y^2 to have the direction of k^α the boundary condition is satisfied identically because the contraction $k\cdot k$ vanishes.

The basic properties of the solution are the following: The gauge-charge density is nonvanishing and the gauge field is nonzero also on the boundary. We expect, however, that the gauge charge is trivial because the solution has the characteristics of the free photon, i.e., magnetic and electric fields are orthogonal (identically) and the action vanishes. Thus the charge density should correspond to polarization charge. The charges corresponding to $SO(3)$ isometries have nonvanishing charge densities except the charge corresponding to the rotation $\Phi\to\Phi+\varepsilon$. The 4-momentum of the solution is lightlike. The angular momentum has *vanishing* component in the direction of motion. However, we think this is not a problem because even for the Maxwell field the spin density and thus spin along the direction of motion is vanishing for a single Fourier mode. (One might of course try to add a suitable divergence term to the angular-momentum current in order to get the spin as a surface contribution: note that $F_{\alpha\beta} \neq 0$ on the boundary.)

An interesting feature of the solution is the way the particle and wave properties of the photon merge together: the photon is a 3-manifold moving with the velocity of light and carrying electric and magnetic fields orthogonal to its direction of motion. Now the photons have rather quantal features already at the classical level: in the short-wavelength limit the particle aspects of the photon should be important and in the long-wavelength limit the field aspect of the photon is expected to dominate.

An open question is whether the nonlinearity of the photon field at the classical level (implying, for example, orthgonality of electric and magnetic fields at the microscopic level) is a fatal feature of the model and how the macroscopic (certainly phenomenological) fields should reflect this property.

Note: It is rather straightforward to show that the photon solutions possess the conformal transformations of S^2 (analogous to the analytic maps of the complex plane) as dynamical symmetries. By definition the transformation induces a multiplicative factor to the metric of S^2 and also the quantities $F_{\alpha\beta}, T^{\alpha\beta}, j^\alpha$ change only by certain multiplicative factors and thus the equations of motion hold true for the transformed 4-surface.

6.2. String Solutions

In the following a set of solutions will be derived characterized by the property that the projection of the representative 4-surface to M^4 or to a linear subspace of it is a two-dimensional minimal surface, which also in the string-model describes the propagation of the string.

Let $A \times B$ denote the decomposition of H into a product of the metric subspaces A and B. There are the following possibilities:

(i) $A \times B = M^4 \times S$

(ii) $A \times B = M^3 \times (M^1 \times S)$

(iii) $A \times B = M^2 \times (M^2 \times S)$

Here the symbol M^k denotes a k-dimensional linear subspace of M^4 (the second factor in the decomposition is always spacelike). The solution type has the general form

$$X^4 = X^2 \times Y^2 \subset A \times B \tag{42}$$

Thus X^2 is a two-dimensional surface in A having interpretation as a string and Y^2 is a two-dimensional surface in B. In the case of the open string the boundary of the three-dimensional spacelike section of X^4 has two components (ends of the string) having the topology of Y^2. The generation hypothesis suggests interpretation of the solution as a neutral meson formed by a quark–antiquark pair of the corresponding generation.

The assumptions about the structure of X^4 imply the separability of the metric, of the second fundamental form, and of the energy-momentum tensor:

$$g_{\alpha\beta}=g_{\alpha\beta}(X^2)+g_{\alpha\beta}(Y^2) \tag{43}$$

$$H^k_{\alpha\beta}=M^k_{\alpha\beta}(X^2)+S^k_{\alpha\beta}(Y^2) \tag{44}$$

$$\begin{aligned} T_{\alpha\beta}=&-\mathrm{Tr}\left(F_\alpha{}^\gamma F_{\gamma\beta}\right)(Y^2)(-g_{\alpha\beta}/4)(Y^2) \\ &\times\mathrm{Tr}\left(F^{\gamma\delta}F_{\gamma\delta}\right)(Y^2)(-g_{\alpha\beta}/4)(X^2)\mathrm{Tr}\left(F^{\gamma\delta}F_{\gamma\delta}\right)(Y^2) \end{aligned} \tag{45}$$

The equations of motion for the coordinates of A are the same as in the string model:

$$g^{\alpha\beta}M^k_{\alpha\beta}=0 \tag{46}$$

X^2 is thus a minimal surface in A. The equations for the coordinates of B express the extremum condition for the magnetostatic energy of the solution:

$$D_\alpha\left(T^{\alpha\beta}h^k{}_\beta\right)-\mathrm{Tr}\left(j^\alpha F^k{}_l h^l{}_\alpha\right)=0 \tag{47}$$

The 4-momentum of the solution can be expressed in the same form as in the string model: now the Regge slope is, however, a dynamical quantity, being proportional to the magnetostatic energy of the solution:

$$P^k=\frac{1}{\alpha}\int\left[g(X^2)\right]^{1/2}g^{0\beta}m^k{}_\beta dx^1 \tag{48}$$

$$\frac{1}{\alpha}=\frac{1}{4}\int\left[g(Y^2)\right]^{1/2}\mathrm{Tr}\left(F^{\alpha\beta}F_{\alpha\beta}\right)dy_1\,dy_2 \tag{49}$$

Thus the interpretation of the solution as a neutral meson gets support also from dynamics.

It is of considerable interest to study the solutions in the case of the minimal choice $S=S^2$. We can note the following facts to be characteristic for this dimension:

(i) The interpretation as magnetic monopoles is possible because the homology charge is the winding number for the coordinate map from the boundary component $\delta X^3{}_i$ to S^2:

$$Q_h = \int (\det s)^{1/2} \varepsilon_{kl} s^k_\alpha s^l_\beta \, dx^\alpha \, dx^\beta \tag{50}$$

This expression, however, represents simply the magnetic flux through the boundary component, as can be seen by using for example the expression for $F_{\alpha\beta}$ in the standard coordinates for S^2.

(ii) Only the "π_0 solution" ($Y^2=S^2$) is of the type $A=M^4$, $B=S^2$. The higher mesons made of higher-generation quarks (genus $g=1,2,\ldots$) and with homological charges $|Q_h|>1$ correspond to the type of solution $A=M^3$, $B=M^1\times S^2$. These mesons correspond effectively to strings in the three-dimensional Minkowski space: the motion is restricted to a plane.

(iii) The Regge slopes for the "higher" mesons are expected to be larger than that of π_0 for the obvious reason.

(iv) The mesons made of quarks corresponding to the nonorientable 2-manifolds are possible only when $A=M^2$ and $B=M^2\times S^2$ because non-orientable 2-manifolds cannot be imbedded in $M^1\times S^2$. Thus only the "yo-yo mode" (Figure 7) is allowed for them and this represents a singular 4-manifold. Thus the minimal model allows only the mesons made of "orientable" quarks.

6.3. Membrane Solutions

The equations of motion also allow solutions which might be called membrane or "soap-film" solutions. Let X^4 have the decomposition

$$X^4 = X^2 \times Y^2 \subset M^3 \times (M^1 \times S)$$

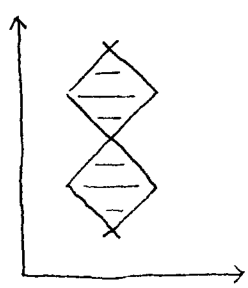

Fig. 7. Illustration of the "yo-yo mode" of the string.

where M^3 is now a spacelike hyperplane in M^4 (M^1 being timelike). Exactly the same arguments as those used for the string solutions show that X^2 must be a minimal surface in M^3 and Y^2 minimizes its electric energy (the equations of motion are formally the same).

The surface X^2 necessarily has boundaries by the well-known theorem for minimal surfaces in Euclidian 3-space (Douglas, 1939). Charges are conserved and are proportional to the area of the surface X^2 (the question whether the boundary conditions can be satisfied is left open). The constant of proportionality is essentially the electric energy per unit area. The mass spectrum is continuous because the solution set possesses scale covariance. The spin of the solution is vanishing. One might interpret the possible quantum-mechanical counterpart as a spinless scalar particle.

6.4. Neutrino Solutions

The simplest type of solution with the spinor degrees of freedom excited is obtained when X^4 is assumed to be a flat, geodesic (second fundamental form vanishes) submanifold of H because the coupling between spinors and surface geometry becomes trivial: as a result we have effectively a massless, free spinor-field theory in X^4.

There are essentially two different ways to choose X^4 when the decoupling requirement is posed. Either

$$\text{(i)}\ X^4 \subset M^4 \times s^0, \qquad \text{where } s^0 \in S$$

or

$$\text{(ii)}\ X^4 \subset M^3 \times S^1 \subset M^4 \times S,$$

where M^3 is a linear subspace of M^4 (with metric having a Minkowskian signature) and S^1 is geodesic in S (a great circle when $S=S^2$).

The equations of motion corresponding to the first possibility are

$$\gamma^\alpha \Psi_\alpha = 0 \tag{51}$$

where the γ matrices are those of M^4. In the second case the equations of motion are

$$\gamma^\alpha \Psi_\psi + 1/R\sigma_\phi \Psi_\phi = 0 \tag{52}$$

where the γ matrices now denote those of a three-dimensional space M^3 and R is the "radius" of S^n.

In the first case the equations represent a massless freely propagating spinor field in $X^4 \subset M^4$. The second case is more complicated: the separable solutions containing the S^1 dependence in the exponential form give rise to a mass spectrum in multiples of $1/R$: of course the lowest state corresponds to a massless particle.

Solutions with compact 3-manifold and satisfying the boundary conditions can be constructed in the following way: Choose $X^{4\ (3)}$ so that it represents a convex region of the 3-space M^3 (2-space M^2) first contracting and then expanding so that the contraction and expansion velocities are asymptotically at least equal to the velocity of light. Give the initial values of the spinor field so that it differs from zero only in a region $V^3 \subset X^3$ ($V^2 \times S^1 \subset X^3 \times S^1$), where X^3 (X^2) represents the convex region when it just begins to expand. Because the spinor field propagates with a finite velocity it never reaches the boundary region and boundary conditions are thus satisfied.

The convexity requirement of the expanding 3- (2-) manifold imply that the boundary of the 3-manifold is either S^2 or $S^1 \times S^1$, corresponding to the cases (i) and (ii), respectively. The generation hypothesis suggests interpretation of the solutions as electronic and muonic neutrinos, respectively. The interpretation is certainly reasonable because the very weak interaction of neutrinos with the other forms of matter is qualitatively understood as a result of decoupling of surface geometry and spinor degrees of freedom for mass-shell neutrinos. Note, however, that the muonic neutrino differs from its electronic counterpart: it has massive excitations: the mass unit is the radius of S^n which is fixed by using the expression for the pionic Regge slope and gauge-coupling as inputs:

$$M = g/\pi \times (\alpha')^{1/2} \tag{53}$$

where the parameter α' is the pionic Regge slope. The first excitation is predicted to have a mass of $\simeq 94\,\mathrm{MeV}$ if the value $1/137$ is used for $g^2/4\pi$.

An important feature of the solutions is the degeneracy resulting from the fact that spinors have 4-components already in the minimal model. In the case of the minimal model the spinors corresponding to the eigenvalue $+1$ (for example) of $i\Gamma_7 = i\gamma_5 \otimes \sigma_3$ can be expressed as a superposition of spinors satisfying

$$i\gamma_5 \Psi = \varepsilon \Psi$$

$$i\sigma_3 \Psi = \varepsilon \Psi \tag{54}$$

with $\varepsilon = \pm 1$.

Note: A solution corresponding to the intuitive picture of the neutrino as a small cylinder moving along the direction of its axis and carrying a spinor field is obtained by assuming for $X^4 \subset M^4$ the same geometry as for the projection of the photon solution to M^4. The boundary conditions are impossible to satisfy (normal should be lightlike everywhere) but different charges are conserved if the nonvanishing Fourier components have wave vectors parallel to the direction of motion. Obviously all neutrino generations are obtained as solutions.

7. CONCLUSIONS AND OUTLOOK

The basic features of the model suggested deserve a short discussion.

(i) Topological Level. Particles (in a loose sense) are identified with three-dimensional submanifolds of some metric space H. The "orbits" of particles correspond to four-dimensional submanifolds of H. Thus the basic structure of the conventional field theories is rejected. The concept of the macroscopic space time is expected to be only an approximate one: the ground state of the theory should correspond to a "porridge" of four-dimensional vacuum surfaces making the propagation of long-range interactions and the concept of macroscopic space-time possible. This expectation gets substantial support provided H is chosen to be of the form $M^4 \times S$ and the action of the theory is chosen to be the Yang–Mills action formally.

The basic hypothesis leads to a rough classification of particles by the boundary topology of the representative 3-manifold. A topological origin for the particle generations is suggested. In the minimal model also the topology of $H = M^4 \times S^2$ becomes important. The boundary components can be classified by their homology-equivalence classes. It is suggested that quarks correspond essentially to homologically (magnetically) charged boundary components.

A generalization for the topological classification of basic vertices of the string model is made. The minimal model affords a nice explanation for the absence of strong interactions of leptons.

(ii) Dynamics. The dynamics is constructed so that formally a standard Yang–Mills theory results. The basic mathematical operation needed is the induction procedure making it possible to construct the metric, the spinor structure and Yang–Mills connection in X^4. In particular the Yang–Mills connection is of completely geometrical origin, being the projection of the Vierbein connection of H to the surface X^4. A unique signature of the choice $H = M^4 \times S$ and the Yang–Mills action is the appearance of the very rich vacuum structure, giving support to the hope that macroscopic space-time indeed results as a "porridge" of vacuum surfaces in H.

(iii) Symmetries and Conservation Laws. The requirement that the isometries of H should be symmetries of the theory leads to the representation of the isometries by parallel translation, i.e., the spinor field is translated along the flow lines of the isometry by parallel translation. This realization, however, works with certain quite restrictive assumptions about the geometry of H. In the minimal model the isometry group contains $SO(3)$ as a factor. The interpretation as "color group" is suggested because the field variables $g_{\alpha\beta}$, $F_{\alpha\beta}$, and Ψ are essentially singlets under it.

The theory is chirally symmetric in a generalized sense, i.e., handedness is defined in H (having even dimension). A possible solution to the so-called chiral problem is suggested based on the possibility of applying the handedness requirement to spinors.

The generalizations of the discrete symmetries C, P, and T are constructed for the minimal model. C operation gets a completely geometric meaning representing reflection in S^2. A rather suprising result is that both C, P, T and CPT(!) are broken when the handedness condition is applied. CP and CT are broken only by boundary conditions and PT is left exact at the classical level.

The gauge currents are identically divergenceless and addition of a suitable total divergence to the action guarantees that the corresponding charges are conserved. Because the gauge field is nonlinear and is subject to boundary conditions, the quantization of the gauge charge already at the classical level is suggested. It is found that a very simple formula for the gauge flux through a boundary component could explain the charges of the known particles. In particular, the charges of mesons could be explained assuming only that the gauge field satisfies the duality condition $F_{\alpha\beta} = \pm F^*_{\alpha\beta}$ on the boundary or has a vanishing normal component.

(iv) The Choice of H. One could argue that the model can be only of a phenomenological significance because the space H appears as a structure not determined by dynamics but is given a priori and thus the model seems to contain a highly subjective element. It is rather surprising to find that the structure of H is to a great extent determined by rather general physical requirements.

Note first that the gauge group of the model is noncompact, consisting of rotations in the tangent space of H. Thus in case of a generic space H we can expect, for example, negative energy densities for the Yang–Mills field. When H has the decomposition $H=M^4\times S$ the gauge group, however, reduces to rotations in the tangent space of S (i.e., the gauge field has components only in the Lie algebra of S) and becomes compact if S is compact.

Assuming this decomposition and accepting the hypothesis that quark confinement is indeed of topological origin one is led to the choice $S=S^2$ as a minimal one. The group $SO(3)$ of S^2 isometries has interpretation as a color group and the fact that all field variables are singlets under $SO(3)$ suggests that the physical states should be $SO(3)$ singlets. The requirement that the energy density of the gauge field is positive and the isometry group of H is the product of the Poincaré group and color group $SO(3)$ leads to the minimal model.

The minimal model with handedness restriction applied to spinors has a formal structure almost that of standard spinor electrodynamics. Now the $U(1)$ gauge field is, however, nonlinear: electric and magnetic fields are identically orthogonal. The photon solutions found, however, suggest already at classical level the description of the photon field using the occupation numbers of the photon modes. An open question is how the phenomenological macroscopic fields should reflect the orthogonality property of the microscopic fields. Already in the minimal model the spinor field has four components, and it remains an open problem whether this means a double degeneracy for the neutrinos of the different generations.

The classical solutions to the equations of motion give strong support to the basic topological ideas of the model. A family of solutions representing neutral mesons as strings with magnetic monopoles at the ends of the string are found. The Regge slope is determined dynamically as magnetostatic energy per unit length of the string. In particular a definite difference between π_0 and other neutral mesons results: π_0 corresponds to a string in M^4 but the higher mesons to strings in M^3. The solutions representing neutrinos are also constructed and the muonic neutrino is predicted to have massive excitations with the mass determined from the Regge slope and gauge coupling.

(v) Open Questions. From the preceding it is clear that there are many interesting problems in the model, which might be approached even in the framework of the classical theory. For example, the problem of whether the gauge charge is quantized and what is the possible charge spectrum could be approached by studying the boundary conditions. A totally untouched problem is of course the construction (or even a proper formulation) of the quantized theory taking into account the strongly geometric nature of the model.

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APPENDIX: SPINOR STRUCTURE FOR $H=M^4\times S$

Assuming that the space S is compact, has spacelike metric, and allows spinor structure, denote by σ_k the Γ matrices of S. The so-called standard representation for the Γ matrices of M^4 is given in the Notation section. The corresponding matrices for the product can be defined as tensor products:

$$\begin{aligned}\Gamma_k &= \gamma_k \times 1, \qquad k=1,\ldots,4\\ \Gamma_{k+4} &= \gamma_5 \times \sigma_k, \qquad k=0,\ldots,\dim S\end{aligned} \tag{A.1}$$

The conjugate spinor $\overline{\Psi}$ can be defined as a spinor

$$\overline{\Psi}=\Psi^*\gamma_4\times 1 \tag{A.2}$$

It is straightforward to verify that the quantity is invariant under local rotations of the group $SO(n+3,1)$ having as infinitesimal generators the matrices

$$\Sigma_{kl}=i/2[\Gamma_k,\Gamma_l] \tag{A.3}$$

When H has even dimension it is possible to define handedness for spinors. This means that we can choose the spinors to be eigen spinors of the matrix $i\Gamma_{n+1}$ defined as

$$i\Gamma_{n+1}=\left[i/(n+4)!\right](-\det h)^{1/2}\varepsilon_{k_1\cdots k_{n+4}}\Gamma^{k_1}\cdots\Gamma^{k_n} \tag{A.4}$$

The eigenvalues correspond to left- and right-handed spinors in H.

The vierbein connection is determined from the requirement of covariant constancy of Γ matrices and has the representation

$$A_k=A_k^{mn}\Sigma_{mn} \tag{A.5a}$$

$$A_k^{mn}=\tfrac{1}{4}\{\Gamma^m, D^n\Gamma_k\} \tag{A.5b}$$

The curvature form of the vierbein connection is expressible using the curvature tensor of H

$$F_{kl} = \tfrac{1}{2} R_{klmn} \Sigma^{mn} \tag{A.6}$$

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